UNDERSTANDING NUMBERS

A new theoretical foundation for mathematics is introduced which explains numbers and numeric operations. In addition to providing a basis for reasoning about what is known, this new theory provides for the construction and definition of new numbers. This foundation secures long sought after stability for mathematics which in turn contributes to the integrity of scientific knowledge.

by Sally Seaver

> "The disagreements concerning what correct mathematics is and the variety of differing foundations affect seriously not only mathematics proper but most vitally physical science.
>
> ...
>
> "The loss of truth, the constantly increasing complexity of mathematics and science, and the uncertainty about which approach to mathematics is secure have caused most mathematicians to abandon science. With a 'plague on all your houses' they have retreated to specialties in areas of mathematics where the methods of proof seem to be safe."[1]

Sally Seaver • PO Box 51159 • Boston, MA 02205-115

1 *Mathematics, The Loss of Certainty* by Morris Kline, Oxford Press: Oxford. 1980. p. 7

Everyone has some experience with numbers and basic arithmetic. Few people, however, are familiar with the academic struggle to define the underlying theory that explains mathematics. Experience has shown that in order to understand mathematics and be certain about what is true, we must know the core assumptions and atomic concepts that provide the foundation for mathematical development. Previous attempts to determine the appropriate theoretical foundation have failed.[1] This paper supplies a new foundation for mathematics to be assessed and tested.

Mathematics appears to be the most trustworthy of scientific disciplines. People use mathematics productively every day. How much impact does a theoretical foundation have?

Something is determined to be true in mathematics by proof via deductive reasoning, rather than by experiments or observation. Proof requires the use of authoritative definitions and standard assumptions called axioms. Thus, definitions and axioms have a critical impact on mathematical knowledge. If they change, then proven results can become unproven. An incorrect bias that is sustained through a poor choice of axioms can inhibit discovery. Also, definitions and axoims that are too difficult to master can alienate potential students.

The theoretical foundation of mathematics determines the axioms and elementary definitions from which the rest of math is deductively derived. Thus, a strong foundation provides the means to accomplish productive research and pedagogy. In other words, the ability to answer questions and discover new things, as well as the ability to teach the discipline of mathematics and provide explanations of why things work the way they do, depend on the theoretical foundation.

In addition to its value to society, the theory of math provides the following value to individuals:

- answers to questions such as "what is a number?", "what separates the topic of math from other topics?", "why is 1+1=2?"
- organization of information
- training for disciplined thinking and reasoning
- basis for work in science

Rigorous knowledge begins with mathematics.

Theory of Mathematics [Math]

This paper proposes a theoretical foundation for the number-related part of mathematics, called One/Plus Theory [1/+ Theory].

Outline of the Presentation

- Organization Begins
- Natural numbers
- Operations
- More numbers
- Zero and other special numbers
- Comparing numbers
- Advanced Topics
- Outside the context of math

Please note that the "Standards for Expressing Definitions" box sets forth special notation for expressing definitions, and the "General Definitions and Assumptions" box sets forth my assumptions in addressing you the reader.

Standards for Expressing Definitions

Conventions used in communicating definitions are as follows:

Def — *signals that the definition of a language rule follows.*

Syn — *signals that the definition of a syntax rule follows.*

≡ — **Indicates that the character string or symbol which precedes it is assigned to represent and designate any meaning which matches the standard immediately following it.**

iff — **if and only if**

A character string in brackets, [ ] , indicates an alternate name for the one being defined.

Organization Begins

Before we can begin to work on the foundation of mathematics, we need to address the general setting in which the development takes place.

The building of ideas, concepts and knowledge is accomplished in the respective minds of individuals. And in each scholar's mind, there are subject-focused workspaces: contexts. For example, a scholar's mind might have a context for literature, chemistry, French, ecology, and politics. In particular, a context should be present for each theory that is studied.

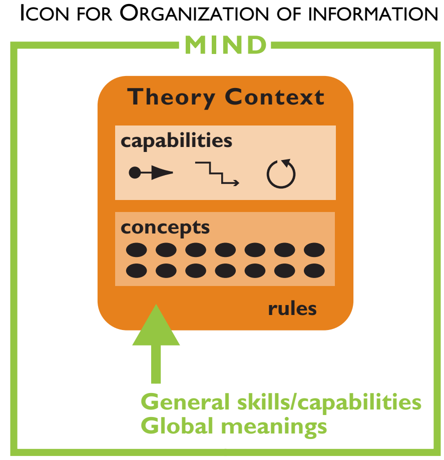
Meanings are developed in a context.

Further organization comes from distinguishing between noun meanings and verb meanings. In this work, a noun meaning corresponds to a concept; a verb meaning corresponds to a means to effect (called a force) that is either a rule or a capability. Rules effect constantly by being present; capabilities effect when they are executed. Each context has its own characteristic verb meanings and noun meanings.

Additionally, each scholar's mind has general capabilities, such as language skills, that can interact with and be applied to different contexts. (My paper, *Principles for Working Together on Knowledge*, addresses general matters of theoretical development more thoroughly.)

The following icon is provided as a guide to help organize these relationships.

Icon for Organization of Information



1 This claim is justified by conclusions in: *The Mathematical Experience* by Philip J. Davis and Reuben Hersh, 1981; and *Mathematics, The Loss of Certainty* by Morris Kline, 1980.

General Definitions and Assumptions

This paper uses the following assumptions, definitions and language practice.

Assumption I
Modern American, written English can be used: a) as a means to communicate, and b) as a known system of symbols and meanings

Assumption II
The words, meanings and language practices defined in this paper supersede or overrule existing definitions.

Rule
The means to effect a restriction.

Capability
The means to effect a result.

Assumption III
For all contexts, the presence of any capability in the context is separate from and independent of the presence of any concept in this context.

Assumption IV
Meanings from English are available for a theory's definition standards, but are not available for construction in a theory context, unless introduced or made present in the theory context by axiom.

Name
A character string which represents and designates (i.e. refers to) meaning.

Standard Language Practice:
Double quotes offset a name when referring to the unique identity of the name itself, both its symbol and meaning. Single quotes offset a name when referring to the unique identity of its character string. Otherwise, names refer to their associated meaning.

= [to equal]
Given names "aa" and "bb", aa = bb if and only if 'aa' refers to the same meaning as 'bb'.

Sequence
A group of things such that each thing either succeeds or precedes every other thing in the group. The *beginning* precedes all others; the *end* succeeds all others.

Do·Step
A construction step comprised of an available capability applied to appropriate, available inputs.

Context of Math

First, we separate math from all other information and ideas. This is accomplished as follows.

Conceive in your mind a place to think that is set apart from existing ideas, concepts, and mental activity. This provides a mental setting that starts out clean and empty. Call this setting the *Context of Math*.

Now determine that one [1] is present in this mental setting. (The meaning of "one" comes from English.) The following statement attempts to provide a sense of what "one" means.

one [1] ≡ a simple concept (no internal structure) with clear boundaries

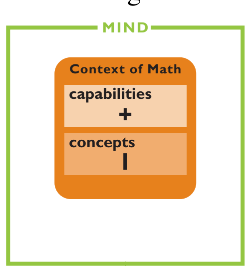
A concept has clear boundaries if we can easily distinguish between what it is and all that it is not.

Using your mental abilities outside the Context of Math, create duplicates of one and move them into the Context of Math. This procedure delivers ones as raw material to this mental workspace. Note that new deliveries of raw material can be ordered at will.

Now, determine that the ability to add is present in this context. (The meaning of "to add" comes from English.) The following definition formalizes the meaning of "to add" in elementary terms.

Definition of to add [+]:
From: 1, 1 (separate and independent)
To: 1 and 1 (together as a whole)

As an capability, + is defined in terms of a direction, *from* an initial state, *to* the resulting state.



The completion of these set-up tasks results in the following axiom being true for the Context of Math.

Basic Axiom

Ones and the ability to add are present.

What is a Number?

Def» number ≡ one and any other concept effected in the Context of Math

To aid in creating concepts in the Context of Math other than one, the following definition is employed.

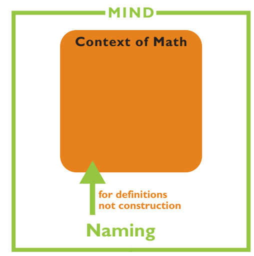
Def» O ≡ a concept in the Context of Math

"O" provides a defining boundary of sorts in that it must be a concept, but at the same time it is completely open for concepts under development.

Naming

What are we going to call each unique number?

Naming is an important activity in the development of knowledge. Names are like holders that capture, separate and maintain the integrity of the meanings being defined. The activity of naming assigns symbols from the context of language to the meanings in the Context of Math.



Over the centuries, certain names have become the standard, by precedent. Names have been chosen for practical reasons: easy to identify, removed from everyday life, free from emotion or culture, and can be written quickly.

1/+ Theory employs both conventional names and new names. Care is taken to provide names that can be spoken as well as written.

Natural Numbers

Math is a discipline like karate or ballet. The theory establishes the proper way to do things and each person who engages in the discipline should expect to exert some effort to master the techniques. Similarly, the learning path for math proceeds from tasks that are fairly easy to tasks that require more mental strength. The first task is to learn the proper way to build concepts called natural numbers.

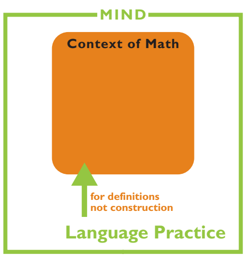
First, before we can begin to build numbers, we need to address a matter of communication.

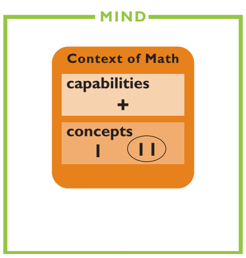
The process of making something usually requires that one step be completed before another step is executed. For example, in making chocolate chip cookies, the batter is mixed together before the cooking or heating process is executed.

The term do·step will be used to refer to a single basic construction step. And, the following language practice will be used to specify the order of steps.

Definition of a language practice:

A do·step that is enclosed by parens, ( ), or brackets, [ ], is executed prior to other do·steps referred to by characters outside of the respective parens, or brakcets.





Building begins with what is available so far: 1 and +. Take a one and add one: 1+1.

Can we add 1 to 1+1? how does the current development justify the use of "+" on the new math objects that get built in the Context of Math?

a. By Assumption III, the meaning and identity of "+" is independent of inputs 1,1 and output 1+1

b. one and one together as a whole has the same key property that one does, namely that it is a concept with a clear boundary; i.e. we can easily distinguish between the whole composed of 1 and 1 and everything else

c. when we try applying + to the concept 1+1 and 1 it works; i.e. we can conceive of what is meant by (1+1)+1, the result makes sense and has meaning

"And" can make a whole (meaning) from any list of unconnected concept meanings. Therefore, going forward, it will be taken for granted that the ability to add can be applied to any concepts present in the Context of Math.

The following indicate some unique concepts that can be effected by adding ones together:

$([[([[(1+1)+1]+1)+1]+1)+1]+1)+1$

$([(1+1)+1]+1)+1$

$([[([[(1+1)+1]+1)+1]+1)+1$

Building unique concepts this way can go on indefinitely. The following definition provides the name and standard used to identify these concepts.

Def» *natural number* [natnum] ≡ 1, and if ❍ is a natural number, then ❍+1.

This definition specifies that a natural number [natnum] is identified by how it was made, i.e. by adding ones together. Some unique natnums are assigned names as follows.

Def» $2 \equiv 1+1$.

Def» $3 \equiv (1+1)+1$.

Def» $4 \equiv [(1+1)+1]+1$.

Def» $5 \equiv ([(1+1)+1]+1)+1$.

Def» $6 \equiv 5+1$.

Def» $7 \equiv 6+1$.

Def» $8 \equiv 7+1$.

Def» $9 \equiv 8+1$.

Def» $10 \equiv 9+1$.

Names for other natural numbers are assigned via the standard naming schema which readers are familiar with, e.g. 11, 23, 101, etc. The box entitled "Base-Ten Names for Common Numbers" contains instructions for this naming schema.

Base ten names provide a way to name a sequence of things that has no end. Much energy and attention is spent during early childhood development mastering this naming schema. Thus, in other cases, to come, where a naming schema is needed for an unending sequence of meanings, Base Ten names will be used in conjunction with other symbols rather than invent a new system of unending names.

The following definitions each provide a short name to represent and designate a natural number, any natural number.

Def» $x \equiv$ natural number.

Def» $y \equiv$ natural number.

Def» $z \equiv$ natural number.

Def» $n \equiv$ natural number.

Def» $m \equiv$ natural number.

Base-Ten Names For Counting Numbers

Def» *Digit list* ≡ '0, 1, 2, 3, 4, 5, 6, 7, 8, 9'

To assign a base-ten name:

- Conceive of a sequence of characters such that the sequence has an end but no beginning and each character in the sequence is '0'. Call this character string [char.str] the 0-char·str.

  0-char·str: …00000000

- Change this 0-char·str to a modified 0-char·str so that each modified 0-char·str is assigned to a single, specific counting number as follows.

  Change '0' in the last character position to '1' […00000001] and assign this name to one.

  …00000001 ↔ one

  For the next counting number in the canonical sequence, x+1, assign the char·str which is built from x's modified 0-char·str by doing the following:

  a) change the character in the last character position to the next, subsequent character in the Digit List (e.g. …0001 → …0002)

  b) if this change requires the next character in the Digit List subsequent to '9', in any character position within the modified 0-char·str, then use '0' in this character position and change the character in the preceding character position to the next, subsequent character in the Digit List(e.g. …0019 → …0020)

  For example:

  x: …0098
  x+1: …0099
  (x+1)+1: …0100

- The base-ten name for each counting number is the modified 0-char·str assigned to it above without the zeros that precede the first non-zero character.

Following are some numbers named this way:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84,…

Definition of a language practice:

Different names that each refer to the same meaning provide independent references to the same meaning.

For example, '2' can be substituted for 'x' and '7' can be substituted for 'y' in any statement which contains both 'x' and 'y', such as: "x+y = z" (2+7 = z) or "x and y are natnums" (2 and 7 are natnums).

Some Properties of "To Add"

Previously, a theory of math was attempted through Peano's Axioms. In that system, the properties of + were assumed as axioms. Here they are derived through reasoning.

I. Commutative Property of +

The result produced by executing + is independent of the order of inputs. In other words: for all x and y

x+y = y + x

"x+y" means x and y (together as a whole) by definition [def] of "+." Similarly, "y+x" means y and x (together as a whole). "x and y" means the same thing as "y and x" due to how "and" works in natural language (i.e. due to the def of "and"). Thus, x+y = y + x by def. of "=".

This same reasoning applies to any number, not just natnums. So for all numbers ❍ and ❑: ❍ + ❑ = ❑ + ❍.

II. Associative Property of +

The result produced by multiple executions of + is independent of the sequence of these executions. This is formally asserted as follows: for all x, y and z, (x+y)+z = x+(y+z)

This is true because "(x+y)+z" refers to the whole meaning composed of x, y, and z and so does "x+(y+z)," by the def of "+." Thus, (x+y)+z = x+(y+z) by def. of "=".

The same reasoning applies to any number, not just natnums. So for all numbers ❍, ❑, ✰:

(❍ + ❑) + ✰ = ❍ + (❑ + ✰)

III. In many cases, the same result can be produced by different construction steps. This is shown by example.

The same result can be effected by different do·steps

			REASON
7+5	=	(6+1) + 5	by def of "7"
	=	6 + (1+5)	by Assoc Prop of +
	=	6+ 6	by def of "6"

Verifying which results of adding are the same by using the respective definitions of natural numbers can be cumbersome. This is why memorization of addition tables is useful.

Def» *Sum* ≡ result of to add.

Def» *Difference* ≡ result of to subtract.

Standard Sequence of Natural Numbers

Just as letters of the alphabet have a standard sequence: a, b, c, d, …etc., natnums are assigned a standard sequence. This sequence will be called *the canonical sequence of natural numbers* and it is defined as follows.

Def» *Canonical sequence of natural numbers* ≡ natural numbers such that 1 precedes all other natural numbers; and given z is a natural number, then z+1 succeeds z.

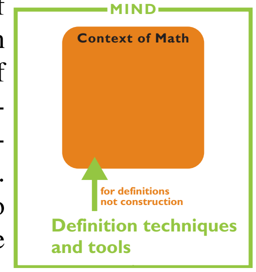
Consequently, a partial list of natural numbers in canonical sequence is: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, …. This sequence is used for many tasks including the task of counting, a topic which will be covered later, outside the Context of Math.

A subtle distinction regarding a sequence of numbers is as follows. The sequential ordering is a type of rule imposed on the concepts that reside in the Context of Math. The sequence as a concept, however, does not reside in the Context of Math since it is created using the ability to conceptualize. The ability to conceptualize is not a math capability, rather it is a general mental capability, and is therefore not available for construction in the Context of Math (per Assumption IV).

Now, consider what is present in the Context of Math so far: some ones, some natnums and the ability to add. How many ones are present? This is not an issue. We are not counting. And we only need enough ones or numbers to complete desired construction steps.

Operations

Definition Techniques and Tools



The next stage of development in the Context of Math is the definition of capabilities other than +. In order to do this, we need the following definition techniques:

- fill-in-the-blank method
- template method
- the ability to patternmatch [#]

Fill-In-The-Blank

Try the following: conceive of a concept, ❍, such that ❍+5 = 3.

This concept is not a natural number since it is not constructed by adding ones together. This concept is defined or separated from the environment provided by the Context of Math using the known meanings of: '5', '3', '+' and '='. Learning the meaning of a new concept this way is like learning the meaning of a new word from how it is used in a sentence. The definition standard of this new concept is: the concept which when added to 5 equals 3.

This method of definition uses the substance of known meanings to create a boundary around a hole such that the new meaning is the one that fits in the hole. Context is critical for providing a setting in which the boundary of the hole makes sense and conceptual substance is present to fill in the blank hole.

This method of definition will be called the *fill-in-the-blank method.*

Now, try using y and x instead of 5 and 3, respectively. Conceive of ❍, such that ❍+y = x. This sets up the means to define concepts—a concept for each unique pair of natural numbers x and y. When a natural number is substituted for x and a natural number is substituted for y, then a definition standard is provided which in essence defines or determines a concept.

Construction Templates

Patterns are used throughout human industry to make things. For example, "copy that letter but address it to the other Senator;" or "make my house like that blueprint."

Now try the following: follow the pattern of construction steps given by (1+1)+1, but use 7 instead of 1: (7+7)+7. Completion of this alternative construction process also effects a concept, 21. The master pattern in this case is the sequence of do·steps used to effect 3: 3 = (1+1)+1.

A master pattern is called a *template*. This method of definition will be called the *template method*.

Now, follow the pattern of construction steps given by (1+1)+1, using x instead of 1: (x+x)+x. This sets up the means to effect many concepts—a concept for each unique natural number x. When a natural number is substituted for x, then execution of the corresponding construction process effects a concept.

Numbers as Construction Patterns

The following rule establishes a standard for using a number to specify a pattern that can be used as a template.

> *Numbers as Templates*
>
> The pattern determined by one is one. The pattern determined by any other number is given by its definition, as it is first defined within the Context of Math.

Thus, the pattern or template, given by a natural number is the sequence of +1 do·steps that were used to effect it according to definition. For example: (1+1)+1 is the template determined by 3, and ([(1+1)+1]+1)+1 is the template determined by 5.

If substitution is made for 1, then it is made for each instance of 1 in the sequence of construction steps. Similarly, if substitution is made for +, then the same substitution is made for each instance of + in the sequence of construction steps.

This convention provides the means to communicate a pattern that can be used as a template in the Context of Math efficiently.

To Patternmatch

How does a pattern of do·steps compare with another?

The instinct to compare is strong in a mathematician. Many language tools and theorems have been devised for this purpose. Observe that the verb to equal [=] provides the means to communicate a comparison of *meanings*, e.g. 3+3 = 6. To equal means that two different names refer to the same meaning. Similarly, we want the means to compare *patterns*. We want the ability to communicate when a pattern is the same as another pattern. For example, consider these patterns:

1	+	1	= 2
25	+	25	=50

(1	+	1)	+	1	= 3
(9	+	9)	+	9	=27

((4	+	4)	+	4)	+	4	=16
((1	+	1)	+	1)	+	1	= 4

How do we talk about the pattern of 1+1 being the same as 25+25? Or how do we express that the pattern of 27 built from nines is the same as the pattern of 3 built from ones? This challenge is addressed by the following language practice and the definition of the ability *to patternmatch*.

Definition of a language practice

> Given a whole, W, parts of the whole named in a list, 'partA,...,partZ', then the list enclosed by braces following the name of the whole, 'W{partA,...,partZ}' indicates: the whole as a pattern such that the parts are open for substitution.

Examples: "Pizza{pepperoni}" refers to a pizza such that pepperoni is a part of the pizza. "Form{name, address, phone number}" means a form such that part of the form is a place for someone's name, a part of the form is for someone's address, and a part is for someone's phone number.

The list of parts following the whole does not need to be exhaustive. Naming a whole and some (or all) of its parts, provides a way to look at the whole as a pattern. We can then conceive of substituting alternates. Some different part can be substituted for the original part. For example, pizza{pepperoni} indicates that a pizza is the whole the pepperoni is free for substitution; so pizza{pepperoni} matches the pattern of pizza{mushroom}. Given form{name, address, phone number}, we have a framework for comparing form{Jane Doe, 123 Main St, 313-555-1000} to form{Bob Smith, 456 Elm St, 414-555-9999}.

Due to the Numbers as Do·Step Templates convention, 3{1,+} indicates 3 as a pattern such that 1 and + are open for substitution.

In the following, let 'A,…,Z' be a list such that each name on the list refers to a part in wholeA and let 'aa,…,zz' be a list such that each name on the list refers to a part in wholeB.

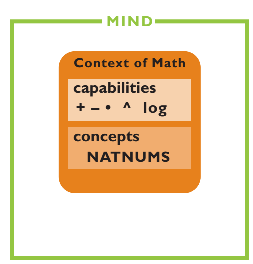
Def» WholeA{A,…,Z} *patternmatches* [#] wholeB{aa,…,zz} iff wholeA{A,…,Z} = wholeB{A,…,Z} and wholeA{aa,…,zz} =wholeB{aa,…,zz}

From the previous examples: 2{1} # 50{25}, 3{1,+} # 27{9,+}, and 16{4,+} # 4{1,+}.

Comparing *patterns* will be used to define math operations. To further aid in this endeavor, the following definition is employed.

Def» *DS* ≡ do·steps in the Context of Math

Some Familiar Capabilities



Following is the proper way (according to 1/+ Theory) to define some familiar operations. These capabilities are established through the definition techniques described above. The pattern of definition implemented below will be used subsequently to define operations that are new to the literature of mathematics.

Def» - [to subtract] ≡
From: x, y
To: ❍ such that ❍+y = x

Syn» x-y

This is the usual ability to subtract that readers are familiar with. In some cases, x-y equals a natural number, e.g.

4-2, and in some cases, x-y does not equal a natural number, e.g. 1-10.

In the past, to subtract has been defined as the ability "to take away" a number from another number. It has also been defined by academic formalists as a mapping or one-to-one association of ordered pairs (x,y) with the number which results from x-y: [(x,y), (x-y)]. In this mapping scenario, capabilities do not exist. Instead, the role of a capability is weakly described by a set whose members are ordered pairs, (input, output). This set of ordered pairs is itself a concept (not a capability). Thus, the 1/+ Theory approach is different from the formal set-theoretic approach.

Now, consider an operation defined as follows:

From: x, y
To: ❍ such that y+❍ = x

in this case, "❍" is on the right of "+." Given the commutative property of addition [Addition Property I], the output given by y+❍ = x is the same as the output specified by ❍+y = x. So this operation equals the operation of subtraction; nothing new is achieved.

The next capability uses the template construction method.

Def» • [to multiply] ≡
From: x, y
To: ❍ such that
❍{x,+} # y{1,+}

Syn» x•y or xy

For example, the +1 construction steps used to effect 4 are [(1+1)+1]+1; and the result of x•4 is the concept effected by executing the steps: [(x+x)+x]+x.

Def» ÷ [to divide by] ≡
From: x, y
To: ❍ such that ❍•y = x.

Syn» x÷y or $\frac{x}{y}$

In some cases, x÷y equals a natural number, e.g. 4÷2, and in some cases, x÷y does not equal a natural number, e.g. 2÷10.

Now, consider an operation defined as follows:

From: x, y
To: ❍ such that y•❍ = x

in this case, "❍" is on the right of "•." Given the commutative property of multiplication for natnums [shown below], the output given by y•❍ = x is the same as the output specified by ❍•y = x. So this suggested operation equals the operation of division; nothing new is achieved.

Commutative Property of Multiplication for Natnums

For all pairs (x,y), x•y = y•x

Justification: y is effected by +1 do·steps by definition. Conceive of a table, such that for each 1 in the template of y, there is a column with 1 as the heading [possible per Assumption I]. In the first row, under the column headings, conceive of x in each cell.

1	1	1	1	…	1	=y
x	x	x	x	…	x	=x•y

Adding the numbers across the row: x+x+x+x+...+x performs the operation given by x•y [by def of •]. Now x is similarly effected by +1 do·steps; so conceive of rows such that the table contains a row for each 1 determined by the template of x. Then populate the table with a one in each cell. For example:

x\y	1	1	1	1	…	1	=y
1	1	1	1	1	…	1	y
1	1	1	1	1	…	1	y
:	:	:	:	:	:	1	:
1	1	1	1	1	…	1	y
=x							=y•x

Adding the y's down the column: y+y+...+y performs the operation given by y•x (by def of •).

x\y	1	1	1	1	…	1	=y
1	1	1	1	1	…	1	y
1	1	1	1	1	…	1	y
:	:	:	:	:	:	1	:
1	1	1	1	1	…	1	y
=x	x	x	x	x	x	x	=x•y

The result of adding all the ones together in the table effects x•y and it effects y•x in keeping with the Commutative Property of +.

Whether the rows are added first and then the columns or the columns and then the rows does not effect the final outcome. Thus, x•y = y•x by definition of "=".

Def» ^ [to raise x to the y power] ≡
From: x, y
To: ❍ such that
❍{x,•} # y{1,+}

Syn» x^y or x^y

For example, x^4 = [(x•x)•x]•x.

Observe that not for all pairs x, y, x^y = y^x, since $3^4 \neq 4^3$. Thus, in this case, the order of ❍ and y in "❍^y = x" matters.

Def» √ [y^{th} root of x] ≡
From: x, y
To: ❍ such that ❍^y = x

Syn» ${}^{y}\sqrt{x}$

Def» log ≡
From: x, y
To: ❍ such that y^❍ = x

Syn» $\log_y x$

In some cases, y√x equals a natural number, e.g. 2√4, and in some cases, y√x does not equal a natural number, e.g. 3√10. Similarly, for some (x,y), $\log_y x$ equals a natural number, e.g. $\log_2 4$, and for some (x,y), $\log_y x$ does not equal a natural number, e.g. $\log_{10} 3$.

In many ways, the above definitions mimic the introduction of the respective operations as they are taught in elementary school, avoiding set-theoretic methods. Nothing new has been achieved so far.

However, the definition of basic operations does not have to stop here. It can continue indefinitely.

Fundamental Capabilities

The basic operations available in the Context of Math are as follows.

Definitions and naming schemas for basic operations:

Def» *First-order op* ≡ $*_n$ such that $*_1$ = + and for $*_{m+1}$
From: x, y
To: ❍ such that
❍{x,$*_m$} # y{1,+}

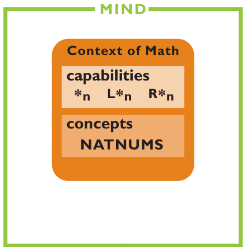
Syn» x $*_n$ y

Def» *L-order op* ≡ L$*_n$ such that
From: x, y
To: ❍ $*_n$ y = x

Syn» x L$*_n$ y

Def» *R-order op* ≡ R$*_n$, such that
From: x, y
To: ❍ such that y $*_n$ ❍ = x

Syn» x R$*_n$ y



Thus, just as the construction of natural numbers can continue indefinitely, so can the definition of basic math operations continue indefinitely.

‘L’ is used in “L-order op” to help remember that ❍ is on the left of the first-order op in the definition standard. Similarly, ‘R’ is used in “R-order op” to help remember that ❍ is on the right.

A convention in conversation is to say “plus” for “+” and “times” for “•.”

For basic operations we need names that are easy to speak.

Definition of a language practice

An alternate way to say “x $*_n$ y” is “x n^{th}-accrue y.” An alternate way to say “x $\text{L}*_n$ y” is “x lop-n y”. An alternate way to say “x $\text{R}*_n$ y” is “x rope-n y.”

Observe the following:

$*_1$ = +	$\text{L}*_1$ = -	$\text{R}*_1$ = -
$*_2$ = •	$\text{L}*_2$ = ÷	$\text{R}*_2$ = ÷
$*_3$ = ^	$\text{L}*_3$ = √	$\text{R}*_3$ = log

Following is an example:

$2 *_4 4 = [(2 *_3 2) *_3 2] *_3 2 = 256$
$256\ \text{R}*_4\ 4 = 2$
$256\ \text{L}*_4\ 2 = 4$

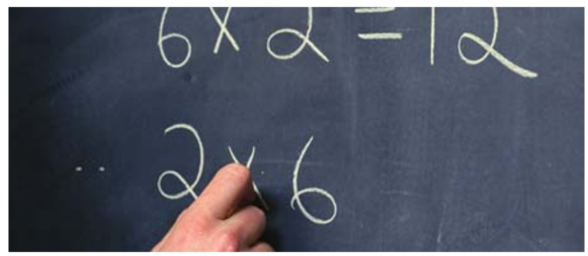
256 $\text{L}*_4$ 3 is not a natnum

256 $\text{L}*_4$ 3 is not a natnum. By definition, 256 $\text{L}*_4$ 3 is the number ? such that ? $*_4$ 3 = 256. Trying natnums in canonical sequence, we get: 1 $*_4$ 3 = 1, 2 $*_4$ 3 = 16, and 3 $*_4$ 3 = 19683. Trying successive numbers in the canonical sequence for ? lead to numbers that come successively after 19683 in the canonical sequence; 256 is before 19683 in the canonical sequence; therefore 256 $\text{L}*_4$ 3 is not a natnum.

Here are some more examples.

Examples Let x=3 and y=4

n	❍	((I	+	I)	+	I)	+	I	base ten name
2	3 $*_2$ 4 =	((3	+	3)	+	3)	+	3	= 12
3	3 $*_3$ 4 =	((3	$*_2$	3)	$*_2$	3)	$*_2$	3	= 81
4	3 $*_4$ 4 =	((3	$*_3$	3)	$*_3$	3)	$*_3$	3	= 7,625,597,484,987



For n = 4, the row provides the do·steps and the result for 3 $*_4$ 4. The first do·step in this row is 3 $*_3$ 3.

(1	+	1)	+	1	= 3
(3	$*_2$	3)	$*_2$	3	= 27

(3 $*_2$ 3) $*_2$ 3, where $*_2$ means to multiply, equals 27. The next do·step is 27 $*_3$ 3.

(1	+	1)	+	1	= 3
(27	$*_2$	27)	$*_2$	27	= 19,683

The next do·step is 19,683 $*_3$ 3.

(1	+	1)	+	1	= 3
(19,683	$*_2$	19,683)	$*_2$	19,683	=

The result is: 7,625,597,484,987.

The steps for 3 $*_n$ 4 where n equals a number in the cannonical sequence after 4 leads to steps where the Base Ten name is too long to write down. 3 $*_5$ 4 and 3 $*_6$ 4, etc. are much more tractable names than the corresponding Base Ten names. Similarly, as x and y are further along in the cannonical sequence, the Base Ten names are impractical.

Operations Take on a Life of Their Own

The basic operations—first-order ops, L-order ops, and R-order ops—have been defined using natnums as inputs. Can we justify using them on other numbers such as 256 $\text{L}*_4$ 3?

Yes, here’s why. The presence of the basic ops in the Context of Math is initially defined using natnums, but by Assumption III, the meaning and identity of the basic operations is independent of natnums. Numbers are noun meanings and operations are verb meanings. When we use verb meanings in a context where they have meaning and obey the proper syntax, then the resulting phrase or sentence has meaning. Just as the ability to add can be applied to new numbers beyond the first known number concept, one, so too can these basic ops be applied to other numbers. Further justification goes beyond the scope of this paper.

Going forward, it will be taken for granted that the basic operations can be applied to any pair of numbers. We reserve the right to establish rules for special handling if and when problems arise from this assumption.

For example, the square-root of 4 can be 2 or -2. The result fo the operation is ambiguous because it does not produce a unique result. But, it is unambiguous in that it specifies a short-list of choices. Experience has shown that taking the square-root of a number x is useful. We just have to use special care to keep track of both options when we work with these $\text{L}*_3$ numbers.

Standard Sequence of Basic Ops

Just as natural numbers were assigned a standard order, so are the basic operations. Let the following organized group of function names be called the *Ops Order Matrix*. The sequence of rows and columns follows from substitution for n in canonical sequence.

Ops Order Matrix

n	$*_n$	$\text{L}*_n$	$\text{R}*_n$
1	$*_1$	$\text{L}*_1$	$\text{R}*_1$
2	$*_2$	$\text{L}*_2$	$\text{R}*_2$
3	$*_3$	$\text{L}*_3$	$\text{R}*_3$
:	:	:	:

Compound Operations

Any operation available in the Context of Math can be used to build other operations. This is accomplished by specifying a sequence of do·steps.

A do·step is a construction step made up of an available capability applied to appropriate, available inputs. For example, the following steps provide the capability to effect a number output from inputs x, y and n.

d) x+y

e) result-of-step-(a) ÷ n

f) $^2\sqrt{\text{result-of-step-(b)}}$

Call this sequence of do·steps *blop*. In this case, executing blop so that it uses 7, 5 and 3 effects the result indicated by $^2\sqrt{((7+5)\div 3)}$, which equals 2.

Def» *function* ≡ a basic or compound operation such that a single execution of the operation results in a single number.

Def» *f* ≡ a function

Def» *g* ≡ a function

The following language practice provides a general way of communicating that a particular function is executed so that it uses the numbers on a given list.

Let '(❍,✰,…,❑)' be a list of names with a beginning and an end such that each name refers to a number concept.

Definition of a language practice:

'*f*(❍,✰,…,❑)' indicates: the result(s) of executing function *f* so that it uses the numbers whose names are on the list "❍,✰,…,❑".

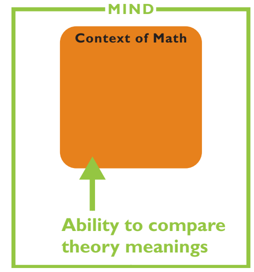
The foundation—whether for a physical structure or a theory—provides support for development. The foundation supports what is built on top of it. We want the foundation to be solid and dependable so that it can support a lot of productive human activity.

For example, blop(7,5,3) = 2.

Blop is a compound operation since it is composed of multiple do·steps. Observe that each of the do·steps contains only a single operation. Each operation is executed in a sequence called a series.

A compound operation can also be composed of multiple do·steps that are to be executed in parallel. These types of compound capabilities are studied in sub-disciplines of math called Linear Algebra and Vector Analysis.

Comparing Operations



Naming, ordering and comparing are standard fare for mathematicians. Here we develop language and standards for comparing different operations.

Domain and Scope

Def» *Domain* of an operation ≡ all that can be affected or used by the operation.

Def» Range of op-A on DomainX≡ all that can be effected by executing op-A so that it uses inputs from DomainX.

In other words, the domain contains the inputs and the range contains the outputs (potential and actual). For example, based on what was delivered to the Context of Math as raw materials, the domain of + was ones, and the range of + on ones was natnums. Then, the case was made that the domain of + is all numbers, and in this case, the range of + on numbers is numbers.

Equal Operations

Given operations op-A and op-B, op-A = op-B if and only if 'op-A' refers the same meaning as 'op-B' [by def in General Definitions and Assumptions box].

The meaning of an operation is a means to effect from..to [on the basis of work in my paper *Principles for Working Together on Knowledge*]. Thus, if both operations have the same from and the same to, then they are equal. In other words, functions that are equal produce the same output when they are applied to the same inputs. They have the same domain and range.

Each Basic Op is Unique

Following is an unsupported claim; proof left to the reader.

Uniqueness of Basic Ops

for all m, n, and z

$*_n = *_m \rightarrow n=m$

$L*_n = L*_m \rightarrow n=m$

$R^*_n = R^*_m \to n=m$

$L^*_n = R^*_m \to n=m$ and (n=1 or n=2)

"$*_n$" ≠ "L^*_m" and "$*_n$" ≠ "R^*_z"

Inverse

Another useful relationship between operations is given as follows.

Def» inverse of function f [f^{-1}] ≡
From: f(❍)
To: ❍

The inverse of a function f, f^{-1}, is like an undo command, because it undoes the effect produced by executing f.

Example: Given $f \equiv 2 \bullet x$ and $g \equiv y \div 2$, then g is the inverse of f since $g(f(x)) = g(2 \bullet x) = (2 \bullet x) \div 2 = x$.

Transparent to Addition

In comparing math operations, the following definition can also be useful.

Def» an operation is *transparent to addition* if and only if it is equivalent to a sequence of do·steps such that the operation in each do·step is +

If an operation is transparent to addition, then when it is applied to natnums, it will effect natnums. [proof left to the reader] For example, • is transparent to addition, but ÷ is not. This is because applying ÷ to ones sometimes results in a non natnum, while executions of + to ones always results in a natnum.

More Numbers

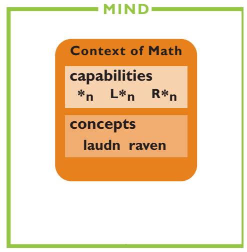
Product Lines

Now all kinds of number concepts can be made in the Context of Math with these basic operations. Different basic operations manufacture different product lines as follows.

Product Lines

numbers constructed to match the following definition standard where x and y are natnums

NAME	DEFINITION STANDARD
Integer	x-y (or x L^*_1 y)
Rational	x÷y (or x L^*_2 y)
Root	$\sqrt[x]{y}$ (or x L^*_3 y)
Log	$\log_y x$ (or x R^*_3 y)
Laudn	x L^*_n y
Raven	x R^*_n y



Just as the construction of natural numbers can continue indefinitely, so can the definition of product lines continue indefinitely.

Laud1 product line = integer product line [laud1s = integers]. Laud2 product line = rational product line [laud2s = rationals]. Product lines for n not equal to 1, 2, or 3, such as Laud4, Laud10, Rave6, Rave50, and Laud100, have not been named or analyzed previously.

We know that the Laud3 and Rave3 product lines include numbers that are not in the Laud2 product line (rationals). I assert that each unique L-order and R-order op produces numbers that are not produced by any other L-order and R-order op. Proof to come later.

Product lines can be branded

One area that these numbers can be put to practical use is cryptography since: a) they involve many operations that are not commutative and b) they help to manage very large numbers.

Bead Numbers

Now, we'll develop numbers analogous to "real" numbers. Note that refering to a line or any other geometric notion is not appropriate for a definition that belongs to the Context of Math.

Def» *Digit list* ≡ '0, 1, 2, 3, 4, 5, 6, 7, 8, 9'

For all n, let k_n be a digit from the digit list; and if there is any ambiguity, 0 is the default digit.

Def» bead number ≡ ❍ such that
❍ = x + d where d is formed by adding a sequence of numbers d_n such that for n=1, $k_1 \bullet (1 \div 10^1) = d_1$ and for n=m, $k_m \bullet (1 \div 10^m) = d_m$ and

Syn» $x.d_1d_2d_3 \ldots d_n \ldots$ such that all "0" digits after the last non "0" digit on the right are dropped.

These numbers are called bead numbers because the construction of these numbers is like stringing multiples of standard units such as 1, 10, 100, 1000,... and 1/10, 1/100, 1/1000... together to make a strand. Only the strand is a sum; and, the number that the name refers to is the result of the sum. Also, these numbers are in the purview of bead arithmetic, calculating techniques involving an abacus.[2]

The syntax rule is the English naming schema for "decimal" numbers which readers are familiar with, e.g. 1.25, 5.089, etc.; $1 \div 3 = .33\overline{3}$

Def» A ≡ bead number

Def» B ≡ bead number

Gigs

The following definition captures all numbers that are created from simple do·steps that involve the basic ops.

Def» *gig* ≡ a **natnum**. a **laudn** number, or a **raven** number is a gig;
and if ❍ and ❑ are gigs, then
❍ $*_n$ ❑, ❍ L^*_n ❑, and ❍ R^*_n ❑
are gigs

Gigs provide a rich universe of concepts for doing mathematical analysis and serving science.

The following definitions each provide a symbol to represent and designate a gig, any gig.

Def» *r* ≡ gig

Def» *s* ≡ gig

Def» *t* ≡ gig

Def» *u* ≡ gig

Def» v ≡ gig

Chinese abacus[3]
Gig numbers are different from bead numbers

Zero and Other Special Numbers

This section introduces special numbers including zero, and one divided by zero.

Zero [0]

Def» *0* [zero] ≡ ❍ such that 1+❍ = 1

We can separate bead numbers (and gigs) into a category called positive

2 http://en.wikipedia.org/wiki/Abacus#Chinese_abacus

3 http://gallery.hd.org/_c/mechanoids/abacus-1-AJHD.jpg.html

numbers and another called negative numbers.

Def» a bead number Bis *positive* iff B= x+d (d given in the definition of bead number).

Def» a bead number, C, is *negative* iff C = 0-(x+d)

Def» a gig is *positive* iff

- it is a natnum, or
- if ❍ and ❑ are positive then, ❍ $*_n$ ❑ is positive for all n, and ❍ $L*_n$ ❑ and ❍ $R*_n$ ❑ are positive for n≠1

Def» a gig is negative iff

- it is not zero, and
- it is not positive

Definition of a language practice:

Given "ω" names a positive number, then the negative number 0-ω is referred to by '-ω'.

This definition formalizes the common practice that readers are familiar with. For example, '-2.03' is used to refer to 0-2.03.

Here is an important property of zero as the behavior of zero given by its definition gets rolled out for all gigs.

Claim: r+0 = r

			REASON
r	=	1+1+…1	by def of x and natnum
r+0	=	(1+1+…+1)+0	same operation to meanings that are the same
	=	(1+1+…)+(1+0)	Assoc Prop of +
	=	1+1+…1	by def of "0"
	=	r	by local def of "r" above

Claim: r•1 = r

Justification: The definition of r•1 is: ❍ such that ❍{r,+} # 1{1,+}. The pattern of 1 is 1. Thus, following the above instructions, r is substituted for 1 resulting in r, so that r•1 = r.

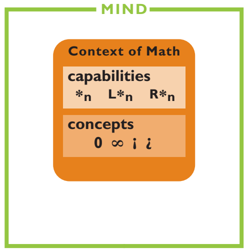
Claim: x•0 = 0

Justification: The definition of x•0 is: ❍ such that ❍{r,+} # 0{1,+}. Let's look at zero as a template pattern. By definition, 0 is ❍ such that 1+❍ = 1. Substituting r for 1 (and + for +): r + ❍ = r. And by the theorem above [r+0 = r], zero satisfies the standard ❍ such that r+❍ = r. Therefore, r•0 = 0.

Claim: $r^0 = 1$

Justification: The concept represented and designated by r^0 is ❍ such that ❍{x,•} # 0{1,+}. By definition, 0 is ❍ such that 1+❍ = 1. Substituting r for 1 and • for +: r • ❍ = r. One satisfies the standard ❍ such that r•❍ = r. Therefore, r^0 =1.

Zeno [∞]



In a departure from mathematics as it has been practiced, consider 1÷0. The definition of 1÷0 is: ❍ such that ❍•0 = 1. By the fill-in-the-blank definition method, this definition is legitimate since the statement "❍•0 = 1" contains a single unknown associated with m and the other names refer to established, known meanings.

Def» ∞ [zeno] ≡ 1÷0

This number is named in honor of a famous mathematician who did significant work related to this concept.

Mathematicians have had a longstanding debate regarding whether or not a completed infinite is present in the Context of Math. This number may be that concept of a completed infinite.

Now, note that either the commutative or associative property of • does not hold for zeno as it does for natnums.

For do·steps involving ∞, order matters

			REASONING
5•∞	=	5• $\frac{1}{0}$	def of "∞"
(5•∞)•0	=	(5• $\frac{1}{0}$)•0	same op to same meaning
(∞•5)•0	≠	(5• $\frac{1}{0}$)•0	different order of inputs for • (not justified for zeno)
∞•(5•0)	≠	5•($\frac{1}{0}$ •0)	different order, (not justified)
∞•0	≠	5•1	by respective defs
1	≠	5	by respective defs

With respect to do·steps in which zeno is present, order matters.

Imaginary Number [¡]

Another special number is $^{2}\sqrt{-1}$. Historically, mathematicians who started to work with this number had trouble conceiving of a number that could be multiplied by itself to produce a negative number. They believed that if it did "exist," then it did not assist with any number-related aspect of reality. Any common number produces a positive number when multiplied by itself, so any number that could be multiplied by itself to produce a negative number must be purely a figment of the imagination.

Mathematicians working with these imaginary numbers determined this. Given a number, φ such that φ•φ = -ω , then φ = B•$^{2}\sqrt{-1}$ where B is a bead number:

φ•φ = (B•$^{2}\sqrt{-1}$)•(B•$^{2}\sqrt{-1}$) = **-1**•B^2 = -ω

This special number, $^{2}\sqrt{-1}$, was called the "imaginary" number. As it turns out, however, this number has proven to be very useful in engineering. It is no more imaginary than any other concept that is defined in the Context of Math.

Def» ¡ [aiye] ≡ $^{2}\sqrt{-1}$

The upside down exclamation point is used to mimic the conventional "i" or "j" while making it different from a letter variable.

Seth [¿]

Another unusual number concept is $\log_1 0$. The definition of $\log_1 0$ is: ❍ such that 1$^{❍}$ = 0. This definition is sufficient due to the Fill-in-the-Blank definition principle.

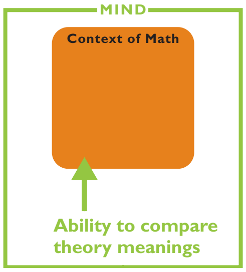
Def» ¿ [seth] ≡ $\log_1 0$

$\log_1 0$ = ❍ such that 1$^{❍}$ = 0. Zero doesn't work because 1^0 = 1 [r^0 = 1 as shown above]. It remains to be shown that 1^r ≠ 0, 1^∞ ≠ 0, and 1$^{¡}$ ≠ 0, so that ¿ is a unique special number. This number is named in honor of my son.

It will be interesting to find out if every unique $L*_n$ and $R*_n$ can be associated with a special number like these.

Comparing Numbers

Whenever we build things, inevitably, we want to compare results of different construction steps. Is this thing bigger, better, more complex, or more interesting than that thing? Also, comparison can provide the means to identify a specific thing, e.g. the tallest building in the city.



An important basis for comparison of numbers is defined as follows.

Def» r > s [r is greater than s] iff r-s = q^+ such that q^+ is a positive, nonzero gig.

Def» r < s [r is less than s] iff s > r.

Def» r ≥ s iff r > s or r = s.

Def» r ≤s iff s > r or s = r.

The following assertion is an important math fact which is used in numerous proofs.

Gig Comparison Result

For all (r,s), r < s, r > s, or r = s.

Justification: for all (r, s), r<s, r>s, or r=s

	REASONING
Either r = s or r ≠ s	tautology: A or not A
r ≠ s iff r-s ≠ 0	r-s = 0 iff r = s; theorem derived from x+0 = x and def of "-"
r-s is a gig	by def of r, s, and gig
r-s ≠0 iff r-s is a nonzero gig; call this nonzero gig q	Steps above and using the ability to name
q is a positive number or it is not	tautology: A or not A
q is positive iff r > s	def of ">"
q is negative iff r < s	def of "<"
r ≠ s iff r > s or r < s	summary of above
Either r = s , r > s, or r < s	First and previous lines

	Do·Steps in a Series	Do·Steps in Parallel	
Inputs	x	x y	
Do·Step	↓ • 2 2x	x y ↓ • 2 ÷ 3 2x y÷3	x y ↓ ÷ 2 • y x÷2 y^2
Do·Step	↓ +1 2x+1	↓ + 2x+(y÷3)	↓ + (x÷2)+y^2
Do·Step	↓ ÷ 2 rational number x.5	↓ • 6 12x+2y	↓ • 6 3x+6y^2
		[a.k.a. vector multiplication]	

$$\begin{bmatrix} x \\ y \end{bmatrix} \left| \begin{matrix} 2 & \frac{1}{3} \\ \frac{1}{2} & y \end{matrix} \right| \cdot 6$$

Just as the natural numbers are assigned a standard sequence, the gigs are assigned a standard sequence.

Def» *Canonical sequence of gigs* [CSRN] ≡ sequence of gig numbers such that any gig succeeds numbers which are less than it and precedes numbers which are greater than it.

A natural question that may come up in comparing groups of numbers is "how many numbers in this group?" Counting or quantifying numbers, however, must be done outside the Context of Math [as addressed below] since a) assigning a quantity is a task using capabilities outside the Context of Math, and b) assigning quantity to numbers involves a form of self-reference.

Magnitude

Sometimes in math, we want to deal with the difference between a number and another number without worrying about whether the smaller number is subtracted from the larger or visa versa. The standard method to accomplish this is to square the difference and then take the square root.

Def» The absolute value of r ≡ $^2\sqrt{(r^2)}$

Syn» ||r||

Sometimes, in a similar way, we want to look at a difference between groups of numbers, say (r, s, t) and (u, v, w). The standard way to accomplish this is to do subtraction pairwise, square each term, add them together and take the square root of the result:
$^2\sqrt{[(r-u)^2+(s-v)^2+(t-w)^2]}$

This kind of numeric difference between groups of numbers is called magnitude.

Def» Given a list of numbers, $(r_1, r_2, ..., r_n)$, and another list of numbers $(s_1, s_2, ..., s_n)$, the *magnitude* of the difference between them is:
≡ $^2\sqrt{[(r_1-s_1)^2+(r_2-s_2)^2+\ldots+(r_n-s_n)^2]}$

Syn» $||(r_1, r_2, ..., r_n) - (s_1, s_2, ..., s_n)||$

Note that the magnitude between a list of numbers, $(r_1, r_2, ..., r_n)$ and zero is:
$||(r_1, r_2, ..., r_n)|| = {}^2\sqrt{[(r_1)^2+(r_2)^2+\ldots+(r_n)^2]}$

Results of Different Capabilities

A common task in mathematics is to investigate whether different capabilities or construction processes (i.e. sequences of do·steps) can produce the same effect. And if they can, in which cases are the results the same and in which cases are they different?

The following provides an example:

Question

Which numbers produced by: $(\log_y x)÷3$, if any, can also be effected by adding ones? Or, in other words, for which pair of natural numbers x, y does $(\log_y x)÷3$ = a natnum?

Answer

If x = y^{3n}, then $(\log_y x)÷3$ = $(\log_y y^{3n})÷3$ = (3n)÷3 = n. So $(\log_y x)÷3$ equals a natural number (the result of adding ones) when the inputs are $(y^{3n}$,y).

Here is a useful definition for looking at the results of different capabilities.

Def» a function *f*(r, t) is *increasing* iff s>t implies *f*(r, s) > *f*(r, t).

Basis Numbers

From the general setting of the mind, connected to general matters of theoretical development—not connected to geometry or math—we can define and talk about dimension. Each dimension is an independent piece of information needed in order to specify a unique target concept. For example, in a spreadsheet, a column and a row specify a unique cell (e.g. in Microsoft Excel, "B5" specifies the cell in the second column, fifth row).

We have learned from complex analysis that expressions of the form A+B¡, where A and B are bead numbers, allow us to caputure numeric relationships for a dimension associated with A and another dimension associated with B. Multiple dimensions recorded within a single sum. This lovely capability (multiple dimensions managed via a single sum) is extended here.

For each unique n: let "B_n" be a name refers to a bead number, and let '$❍_n$' be a name that refers to a number, such that "n" is an index for identifying unique names.

Def» Rockn [$®_n$] ≡ $❍_n$ such that for n=1
$®_1$ = 1 and for n=m
$®_m$ = ❍ such that for all non zero bead numbers A & B, and all natnums i, i < m, **B•❍** ≠ **A•**$®_i$

Def» *Basis number* ≡
$(B_1•❍_1)+(B_2•❍_2)+…+(B_n•❍_n)$

Verify that ∞, ¡ and ¿ are rock numbers. In this case, 1 = Rock1, ∞ = Rock2, ¡ = Rock3, and ¿ = Rock4. Consider the following numbers:

$(B_1•1)+(B_2•∞)+(B_3•\text{¡})+(B_4•\text{¿})$

i.e. $B_1+B_2∞+B_3\text{¡}+B_4\text{¿}$.

Look at this example:
1.23 + 99.9∞ + 703¡ + 40.56¿

$[(1 + (2•(1÷10^1)) + (3•(1÷10^2))) • 1] +$
$[(99 + (9•(1÷10^1)) + (0•(1÷10^2))) • ∞] +$
$[(703 + (0•(1÷10^1)) + (0•(1÷10^2))) • \text{¡}] +$
$[(40 + (5•(1÷10^1)) + (6•(1÷10^2))) • \text{¿}]$

If it was not the case that **B•**$®_n$ ≠ **A•**$®_i$, then the terms could be manipulated so that bead numbers B_n could be combined; they could interact so that they don't maintain their independence. For example, if **B•**¡ = **A•**¿, then

$[(703 + (0•(1÷10^1)) + (0•(1÷10^2))) • \text{¡}] +$
$[(40 + (5•(1÷10^1)) + (6•(1÷10^2))) • \text{¿}]$
=
$[(703 + (0•(1÷10^1)) + (0•(1÷10^2))) • \text{¡}] +$
$[(40 + (5•(1÷10^1)) + (6•(1÷10^2))) • {}^B/_A\text{¡}]$

$= (703 + (40.56• {}^B/_A)) • \text{¡}$

We know that a bead number divided by a bead number is another bead number, and a bead number multiplied by a bead number is also another bead number. Thus, this result is a bead number times aiye. And in this case, the numbers named by B_3 and B_4 did not maintain their independence.

A noteworthy property basis numbers formed from Rock numbers 1, ∞, ¡, and ¿ is that when B_n=0 for all n, then the basis number = 1, since 0• ∞ =1.

Although the concept of a basis is used in Linear Algebra (defined a little differently) the concept of a basis number is new to the literature of math.

■ This concludes construction and definition within the Context of Math

Summary

What has been accomplished so far? In the Context of Math, the basic operations have been defined and established, not just the conventional operations: +, -, •, ÷, ^, √, and log, but many more. All numbers have been built or defined using one and "+."

Language infrastructure for pattern-matching has been established as well as language practices for definition and construction techniques.

New groups of numbers, besides integers, rationals, and traditional irrationals have been identified and categorized in different product lines. The idea of a real number becomes outdated; instead, we have bead numbers and gigs.

Rockn numbers and basis numbers have been introduced which provide even more powerful tools in the support of support of science since they enable the expression of multi-dimensional relationships in a single (linear) sum.

The elementary meanings that provide a basis for comparison were formalized in this context for both operations and numbers.

The Context of Math starts out in a state or condition that is explicitly known. The forces that can effect change are clearly identified. Neither the operations (capabilities) or the concepts in this context violate each other, thus the integrity of their respective defining-limits is preserved. [Proof of this claim goes beyond the scope of this paper.] Therefore, the meanings needed for doing mathematics have been established in such a way that results are reliable and consistent.